

Available online at www.sciencedirect.com

SCIENCE @ DIRECT®

Journal of Functional Analysis 237 (2006) 482–506

JOURNAL OF
Functional
Analysiswww.elsevier.com/locate/jfa

A continuous field of C^* -algebras and the tangent groupoid for manifolds with boundary

Johannes Aastrup^a, Ryszard Nest^b, Elmar Schrohe^{a,*}^a *Institut für Analysis, Universität Hannover, Welfengarten 1, 30167 Hannover, Germany*^b *Department of Mathematics, Copenhagen University, Universitetsparken 5, 2100 Copenhagen, Denmark*

Received 18 July 2005; accepted 2 November 2005

Available online 24 January 2006

Communicated by Alain Connes

Abstract

For a smooth manifold X with boundary we construct a semi-groupoid T^-X and a continuous field $C_r^*(T^-X)$ of C^* -algebras which extend Connes' construction of the tangent groupoid. We show the asymptotic multiplicativity of \hbar -scaled truncated pseudodifferential operators with smoothing symbols and compute the K -theory of the associated symbol algebra.

© 2005 Elsevier Inc. All rights reserved.

Keywords: Manifolds with boundary; Continuous fields of C^* -algebras; Tangent groupoid

Contents

0. Introduction	483
1. The classical case	486
2. Manifolds with boundary	490
3. K -theory of the symbol algebra $C_r^*(T^-X)$	504

* Corresponding author.

E-mail addresses: aastrup@math.uni-hannover.de (J. Aastrup), rmest@math.ku.dk (R. Nest), schrohe@math.uni-hannover.de (E. Schrohe).

0. Introduction

It is a central idea of semi-classical analysis to consider Planck's constant \hbar as a small real variable and to study the relation between systems in mechanics and systems in quantum mechanics by associating to a function $f = f(x, \xi)$ on the cotangent bundle of a manifold the \hbar -scaled pseudodifferential operator $\text{op}_\hbar(f)$ with symbol $f(x, \hbar\xi)$ and analyzing their relation as $\hbar \rightarrow 0$.

For $f \in \mathcal{S}(T^*\mathbb{R}^n)$, for example, a basic estimate states that

$$\lim_{\hbar \rightarrow 0} \|\text{op}_\hbar(f)\| = \|f\|_{\text{sup}}. \quad (0.1)$$

Moreover, given a second symbol $g \in \mathcal{S}(T^*\mathbb{R}^n)$ we have

$$\lim_{\hbar \rightarrow 0} \|\text{op}_\hbar(f) \text{op}_\hbar(g) - \text{op}_\hbar(fg)\| = 0; \quad (0.2)$$

in other words, the map op_\hbar is asymptotically multiplicative.

As both statements concern the asymptotic behavior of pseudodifferential operators, it is somewhat surprising that they can be proven within the framework of continuous fields of C^* -algebras associated to amenable Lie groupoids, more precisely, the C^* -algebra of the so-called tangent groupoid $\mathcal{T}M$, cf. Connes [6, Section II.5].

For a boundaryless manifold M , $\mathcal{T}M$ is constructed by gluing the tangent space TM to the Cartesian product $M \times M \times]0, 1]$ via the map $TM \times [0, 1] \ni (m, v, \hbar) \mapsto (m, \exp_m(-\hbar v), \hbar)$. It has the natural cross-sections $\mathcal{T}M(\hbar)$, $0 \leq \hbar \leq 1$, given by TM for $\hbar = 0$ and by $M \times M \times \{\hbar\}$ for $\hbar \neq 0$.

The basic observation, establishing the link between \hbar -scaled pseudodifferential operators and the tangent groupoid, is the following. In the Fourier transformed picture, the \hbar -scaled pseudodifferential operator $\text{op}_\hbar(f)$ becomes the convolution operator $\rho_\hbar(\hat{f})$ acting by

$$\rho_\hbar(\hat{f})\xi(x) = \frac{1}{\hbar^n} \int \hat{f}\left(x, \frac{x-y}{\hbar}\right) \xi(y) dy, \quad \xi \in L^2(\mathbb{R}^n),$$

and for $\hbar \neq 0$, the mappings ρ_\hbar (or better their generalization to the manifold case) coincide with the natural representations of $C_c^\infty(\mathcal{T}M(\hbar))$ by convolution operators.

The ρ_\hbar , $\hbar \neq 0$, are complemented by the representation π_0 of $C_c^\infty(TM)$ on $L^2(TM)$ via convolution in the fiber which in turn coincides with the natural representation of $C_c^\infty(\mathcal{T}M(0))$.

Now the tangent groupoid is additionally amenable, so that, according to a theorem by Anantharaman-Delaroche and Renault [2], the reduced C^* -algebra $C_r^*(\mathcal{T}M)$, defined as the closure of $C_c^\infty(\mathcal{T}M)$ with respect to the natural representations, and the full C^* -algebra $C^*(\mathcal{T}M)$, i.e., the closure with respect to all involutive Hilbert space representations, are isomorphic.

The crucial fact then is that $C_r^*(\mathcal{T}M)$ is a continuous field of C^* -algebras over $[0, 1]$; the fiber over \hbar is $C_r^*(\mathcal{T}M(\hbar))$. An elegant way to establish the continuity is to show upper semi-continuity and lower semi-continuity separately, noticing that upper semi-continuity is easily proven in $C^*(\mathcal{T}M)$ while lower semi-continuity is not difficult to show in $C_r^*(\mathcal{T}M)$. As both C^* -algebras are isomorphic, continuity follows. For a good account of these facts see [10] by Landsman and Ramazan. The identities (0.1) and (0.2) are then an immediate consequence of the continuity of the field.

In the present paper we consider manifolds with boundary. The analog of the usual pseudodifferential calculus here is Boutet de Monvel's calculus for boundary value problems [3]. In order

to obtain an operator algebra, one cannot work with pseudodifferential operators alone, but has to introduce an additional class of operators, the so-called singular Green operators. The reason is the way pseudodifferential operators act on functions defined on a half space: one first extends the function (by zero) to the full space, then applies the pseudodifferential operator and finally restricts the result to the half space again—one often speaks of truncated pseudodifferential operators. Given two pseudodifferential operators P and Q , the ‘leftover operator’ $L(P, Q) = (PQ)_+ - P_+Q_+$, i.e., the difference between the truncated pseudodifferential operator $(PQ)_+$ associated to the composition PQ and the composition of the truncated operators P_+ and Q_+ associated with P and Q is a typical example of such a singular Green operator. The singular Green operators “live” at the boundary. They are smoothing operators in the interior, while, close to the boundary, they can be viewed as operator-valued pseudodifferential operators along the boundary, acting like smoothing operators in the normal direction.

In the full algebra which consists—at least in the slightly simplified picture we have here—of sums of (truncated) pseudodifferential operators and singular Green operators, the singular Green operators form an ideal.

With this picture in our mind, we construct an analog of Connes’ tangent groupoid for a manifold X with boundary. Our semi-groupoid \mathcal{T}^-X consists of the groupoid $X \times X \times]0, 1]$ to which we glue, with the same map as above, the half-tangent space T^-X , which comprises all those tangent vectors v in points $m \in X$ for which $\exp_m(-\hbar v)$ lies in X for small \hbar (note that this condition is only effective at the boundary of X). As before, we have natural cross-sections $\mathcal{T}^-X(\hbar)$, coinciding with $X \times X \times \{\hbar\}$ for $\hbar \neq 0$ and with T^-X for $\hbar = 0$.

For $\hbar \neq 0$, the operators ρ_\hbar (with integration now restricted to X), are the natural representations of the groupoid $\mathcal{T}^-X(\hbar)$. At $\hbar = 0$ we use two mappings. The first, π_0 is the analog of the above map π_0 . It acts on the tangent space of X by convolution. The second, π_0^∂ , acts on the half-tangent space over the boundary by half-convolution: $\pi_0^\partial : C_c^\infty(\mathcal{T}^-X) \rightarrow \mathcal{L}(L^2(T^+X|_{\partial X}))$ is given by

$$\pi_0^\partial(f)\xi(m, v) = \int_{T_m^+X} f(m, v - w)\xi(m, w) dw.$$

In order to avoid problems concerning the topology of \mathcal{T}^-X , we denote by $C_c^\infty(\mathcal{T}^-X)$ the space of all restrictions of functions in $C_c^\infty(\mathcal{T}\tilde{X})$ to \mathcal{T}^-X ; here \tilde{X} is a boundaryless manifold containing X .

The reduced C^* -algebra $C_r^*(\mathcal{T}^-X)$ is then defined as the C^* -closure of $C_c^\infty(\mathcal{T}^-X)$ with respect to the ρ_\hbar , $\hbar \neq 0$, and π_0, π_0^∂ for $\hbar = 0$. For the full C^* -algebra we use all involutive representations.

We show that $C_r^*(\mathcal{T}^-X)$ is a continuous field of C^* -algebras over $[0, 1]$, where the fiber over $\hbar \neq 0$ is $C_r^*(\mathcal{T}^-X(\hbar)) = \mathcal{K}(L^2(X))$, and the fiber over $\hbar = 0$ is the C^* -closure of $C_c^\infty(\mathcal{T}^-X)$ with respect to π_0 and π_0^∂ .

The proof of continuity is again split up into showing upper semi-continuity and lower semi-continuity. According to an idea by Rieffel [20], lower semi-continuity is established using strongly continuous representations. The basic idea for the proof of upper semi-continuity would be to infer an isomorphism between $C_r^*(\mathcal{T}^-X)$ and $C^*(\mathcal{T}^-X)$ from the amenability of \mathcal{T}^-X . However, as \mathcal{T}^-X is only a semi-groupoid, we make a little detour, and use short exact sequences and the amenability of the tangent groupoids for boundaryless manifolds.

The present study should be seen as a step towards fitting Boutet de Monvel's calculus for boundary value problems into the framework of deformation quantization and groupoids, in the spirit of Connes [6], Monthubert and Pierrot [14], Fedosov [9] Nest and Tsygan [15,16], Nistor et al. [17]. Eventually one could hope to develop an algebraic index theory for these deformations in the spirit of Nest and Tsygan.

The structure of the paper is as follows. In the first section we review the case of boundaryless manifolds. We introduce the basic notions and show how (0.1) and (0.2) are derived with the help of the continuous field of C^* -algebras associated to the tangent groupoid.

We then consider a manifold X with boundary. In order to make the presentation more transparent, we first study the case where $X = \mathbb{R}_+^n = \{(x_1, \dots, x_n) \mid x_n \geq 0\}$. Here all relevant features show up, but computations are easier to perform. We then go over to the general case.

In Section 3 we determine the K -theory of the symbol algebra $C_r^*(T^-X)$. Starting from the short exact sequence

$$0 \rightarrow C_r^*(TX^\circ) \rightarrow C_r^*(T^-X) \rightarrow C_r^*(T^-X)/C_r^*(TX^\circ) \rightarrow 0$$

we show that the quotient can be identified with $C_0(T^*\partial X) \otimes \mathcal{T}_0$, where \mathcal{T}_0 is an ideal in the Toeplitz algebra with vanishing K -theory. In particular, we obtain the isomorphism

$$K_i(C_r^*(T^-X)) \cong K_i(C_0(T^*X)), \quad i = 0, 1;$$

compare with the K -theory of Boutet de Monvel's algebra computed in [12].

The appearance of the Toeplitz operators can be seen as a feature inherent in the geometry of the problem. In fact, the construction of an algebra of pseudodifferential operators on a closed (Riemannian) manifold amounts to the construction of a suitably completed operator algebra, generated by multivariable functions of vector fields and the operators of multiplication by smooth functions.

In the boundaryless case, one can localize to \mathbb{R}^n and reduce the task essentially to defining $f(D)$ for a classical symbol f and $D = (D_1, \dots, D_n)$ with the vector fields $D_j = i\partial_{x_j}$. One convenient way of achieving this is to use the operator families e^{itD_j} and to let

$$f(D) = (2\pi)^{-n} \int \hat{f}(\xi) e^{i\xi D} d\xi$$

with the Fourier transform \hat{f} of f and $\xi D = \xi_1 D_1 + \dots + \xi_n D_n$. Note that the use of the $e^{i\xi D}$ is purely geometric and only relies on the fact that vector fields integrate to flows.

On a manifold with boundary, one will have vector fields transversal to the boundary which do not integrate to flows. In this case, one has two possibilities: the first is to restrict the class of admissible vector fields to those which do integrate. This is a basic idea in the pseudodifferential calculi introduced by Melrose [13], see also Ammann et al. [1].

In Boutet de Monvel's calculus, on the other hand, transversal vector fields are admitted. After localization to \mathbb{R}_+^n , we may focus on D_n . One of the functions one would certainly like to define is the Cayley transform (recall that the Cayley transform $C(A)$ of an operator A is given by $C(A) = (A - i)(A + i)^{-1} = 1 - 2i(A + i)^{-1}$).

Now it is well known that the Cayley transform $C(A)$ is an isometry, and that it is a unitary if and only if A is selfadjoint. As there is no selfadjoint extension of D_n , its Cayley transform will be a proper isometry. Hence by a theorem of Coburn [4,5], the algebra generated by it (which becomes part of the calculus), is the Toeplitz algebra.

While the pseudodifferential calculus for closed manifolds is commutative modulo lower order terms, this calculus is not. From a geometric point of view, the resulting algebra can thus be seen as a noncommutative completion of the manifold with boundary.

Remark on the notation. A variety of representations naturally comes up in this context. In order to distinguish their origin, we will apply the following rule. Representations related to the groupoid structure are denoted by π (possibly indexed), asymptotic pseudodifferential operators by ρ_{\hbar} and the asymptotic Green operators (introduced in Section 2) by κ_{\hbar} .

1. The classical case

Groupoids

A groupoid G is a small category where all the morphisms are invertible. We will denote by $G^{(0)}$ the set of objects in G and by $G^{(1)}$ the set of morphisms. We will also call $G^{(0)}$ the base and the elements in $G^{(1)}$ the arrows. On $G^{(1)}$ there are two maps r, s into $G^{(0)}$. The first map, r , is the range object of a morphism and the second, s , the source. For $x \in G^{(0)}$ we define $G^x = r^{-1}(x)$ and $G_x = s^{-1}(x)$. There is an embedding ι of $G^{(0)}$ into $G^{(1)}$ given by mapping an object to the identity morphism on this object. Furthermore we define $G^{(2)}$ to be the subset of composable morphisms of $G^{(1)} \times G^{(1)}$.

1.1. Definition. A Lie groupoid G is a groupoid together with a manifold structure on $G^{(0)}$ and $G^{(1)}$ such that the maps r, s are submersions, the map ι and the composition map $G^{(2)} \rightarrow G^{(1)}$ are smooth.

To a given smooth manifold M without boundary there are associated two canonical Lie groupoids. The first is the tangent bundle TM of M . The groupoid structure is given by

$$\begin{aligned} G^{(0)} &= M, & G^{(1)} &= TM, \\ r(m, X) &= m, & s(m, X) &= m, \\ (m, X) \circ (m, Y) &= (m, X + Y). \end{aligned}$$

The second one is the pair groupoid $M \times M$ with

$$\begin{aligned} G^{(0)} &= M, & G^{(1)} &= M \times M, \\ r(m_1, m_2) &= m_1, & s(m_1, m_2) &= m_2, \\ (m_1, m_2) \circ (m_2, m_3) &= (m_1, m_3). \end{aligned}$$

Both are clearly Lie groupoids.

1.2. Haar systems. A smooth left Haar system on a Lie groupoid is a family of measures $\{\lambda^x\}_{x \in G^{(0)}}$ on G with $\text{supp } \lambda^x = G^x$ which is left invariant, i.e., $\gamma(\lambda^{s(\gamma)}) = \lambda^{r(\gamma)}$, and for each $f \in C_c^\infty(G^{(1)})$, the function on $G^{(0)}$ defined by

$$x \mapsto \int f d\lambda^x, \quad f \in C_c^\infty(G^{(1)}),$$

is smooth. In [10, Proposition 3.4], it is proven that all Lie groupoids possess a smooth left Haar system. Similarly, a right Haar system $\{\lambda_x\}$ is given by $\lambda_x = (\lambda^x)^{-1}$.

1.3. Definition. A Lie groupoid G with a smooth left Haar system λ^x is called topologically amenable if there exists a net of nonnegative continuous functions $\{f_i\}$ on $G^{(1)}$ such that:

- (i) for all i and for $x \in G^{(0)}$, $\int f_i d\lambda^x = 1$;
- (ii) the functions $\gamma \mapsto \int |f_i(\gamma^{-1}\gamma') - f_i(\gamma')| d\lambda^{r(\gamma)}(\gamma')$ converge uniformly to zero on compact subsets of $G^{(1)}$.

It is easy to verify that the two groupoids TM and $M \times M$ are topologically amenable.

1.4. Connes' tangent groupoid. Let M be a smooth manifold. Connes tangent groupoid TM is a blow up of the diagonal in $M \times M$. More specifically, let $TM = TM \cup (M \times M \times]0, 1])$ as a set. The groupoid structure is just the fiberwise groupoid structure coming from the groupoid structure on TM and $M \times M$. The manifold structure on $M \times M \times]0, 1]$ is obvious. We next glue TM to $M \times M \times]0, 1]$ to get a manifold structure on TM . We choose a Riemannian metric on M and define the charts

$$TM \times [0, 1] \supseteq U \ni (m, v, \hbar) \mapsto \begin{cases} (m, v) & \text{for } \hbar = 0, \\ (m, \exp_m(-\hbar v), \hbar) & \text{for } \hbar \neq 0, \end{cases}$$

where \exp_m denotes the exponential map, U is an open neighborhood of $M \times \{0\} \subset TM \times \{0\}$, and M is embedded as the zero section.

Here, $G^{(0)} = M \times [0, 1]$. For $\hbar \neq 0$ and $x = (\tilde{m}, \tilde{\hbar}) \in G^{(0)}$, we have $G^x = \{(\tilde{m}, m, \tilde{\hbar}) : m \in M\}$; for $x = (\tilde{m}, 0)$, $G^x = T_{\tilde{m}}M$. Fixing the measure μ on M induced by the metric, we obtain a Haar system $\{\lambda^x\}_{x \in G^{(0)}}$ by $\lambda^{(\tilde{m}, \tilde{\hbar})} = \hbar^{-n} \mu$, $\hbar \neq 0$; for $\hbar = 0$, we let $\lambda^{(\tilde{m}, 0)}$ be the measure on $T_{\tilde{m}}M$ given by the metric.

This makes TM a Lie groupoid, see [6,10].

1.5. C^* -algebras associated to groupoids. Let G be a Lie groupoid with a smooth left Haar system λ . On $C_c^\infty(G^{(1)})$ we define a $*$ -algebra structure by

$$(f * g)(\gamma) = \int_{G^{s(\gamma)}} f(\gamma\gamma_1)g(\gamma_1^{-1}) d\lambda^{s(\gamma)}(\gamma_1), \quad (1.1)$$

$$f^*(\gamma) = \overline{f(\gamma^{-1})}. \quad (1.2)$$

There are involutive representations π_x , $x \in G^{(0)}$, of this $*$ -algebra on the Hilbert spaces $L^2(G_x, \lambda_x)$ given by

$$\pi_x(f)\xi(\gamma) = \int_{G^x} f(\gamma\gamma_1)\xi(\gamma_1^{-1}) d\lambda^x(\gamma_1), \quad \xi \in L^2(G_x, \lambda_x). \quad (1.3)$$

See Renault's book [19] for details.

1.6. Definition. The full C^* -algebra $C^*(G)$ of a groupoid is the C^* -completion of the $*$ -algebra $C_c^\infty(G^{(1)})$ with respect to all involutive Hilbert space representations.

The reduced C^* -algebra $C_r^*(G)$ of G is the C^* -completion of $C_c^\infty(G)$ with respect to the representations (1.3).

Note that, by universality, we have a quotient map from $C^*(G)$ to $C_r^*(G)$.

1.7. Remark. Although the construction of the $*$ -algebra structure on $C_c^\infty(G^{(1)})$ and the representations (1.3) use a smooth Haar system, the algebra is independent of the choice. See [11] for a detailed exposition.

1.8. Example. For the tangent bundle TM of a manifold, the space G_m is just $T_m M$ and the representation is

$$\pi_m(f)\xi(v) = \int_{T_m M} f(m, v-w)\xi(w)dw, \quad \xi \in L^2(T_m M).$$

By Fourier transform in each fiber $T_m M$, the C^* -algebra $C_r^*(TM)$ becomes isomorphic to $C_0(T^*M)$, the continuous functions on T^*M vanishing at infinity.

The importance of topological amenability lies in the following result from [2].

1.9. Proposition. When G is topologically amenable the quotient map from $C^*(G)$ to $C_r^*(G)$ is an isomorphism.

Continuous fields and \hbar -scaled pseudodifferential operators

1.10. Definition. A continuous field of C^* -algebras $(A, \{A(\hbar), \varphi_\hbar\}_{\hbar \in [0,1]})$ over $[0, 1]$ consists of a C^* -algebra A , C^* -algebras $A(\hbar)$, $\hbar \in [0, 1]$, with surjective homomorphisms $\varphi_\hbar : A \rightarrow A(\hbar)$ and an action of $C([0, 1])$ on A such that for all $a \in A$:

- (i) the function $\hbar \mapsto \|\varphi_\hbar(a)\|$ is continuous;
- (ii) $\|a\| = \sup_{\hbar \in [0,1]} \|\varphi_\hbar(a)\|$;
- (iii) for $f \in C([0, 1])$, $\varphi_\hbar(fa) = f(\hbar)\varphi_\hbar(a)$.

1.11. Theorem. For the tangent groupoid TM we define $TM(0) = TM$ and $TM(\hbar) = M \times M \times \{\hbar\}$ for $\hbar \neq 0$. The pullback under the inclusion $TM(\hbar) \hookrightarrow TM$ induces a map $\varphi_\hbar : C_c^\infty(TM) \rightarrow C_c^\infty(TM(\hbar))$ which extends by continuity to a surjective $*$ -homomorphism $\varphi_\hbar : C_r^*(TM) \rightarrow C_r^*(TM(\hbar))$. The C^* -algebras $A = C_r^*(TM)$ and $A(\hbar) = C_r^*(TM(\hbar))$ with the maps φ_\hbar form a continuous field over $[0, 1]$.

Proof. Together with the amenability of TM and Proposition 1.9 this is immediate from [10, Theorem 6.4]. \square

1.12. \hbar -Scaled pseudodifferential operators. For $0 < \hbar \leq 1$ define $\rho_\hbar : C_c^\infty(T\mathbb{R}^n) \rightarrow \mathcal{L}(L^2(\mathbb{R}^n))$ by

$$\begin{aligned} \rho_\hbar(f)\xi(x) &= \int f(x, w)\xi(x - \hbar w)dw \\ &= \hbar^{-n} \int f\left(x, \frac{x-w}{\hbar}\right)\xi(w)dw, \quad \xi \in L^2(\mathbb{R}^n). \end{aligned} \quad (1.4)$$

We complement this by the map $\pi_0 : C_c^\infty(T\mathbb{R}^n) \rightarrow \mathcal{L}(L^2(T\mathbb{R}^n))$

$$\pi_0(f)\xi(x, v) = \int f(x, w)\xi(x, v - w) dw. \quad (1.5)$$

1.13. Remark. (a) We can define $\tilde{\rho}_h : C_c^\infty(T\mathbb{R}^n) \rightarrow \mathcal{L}(L^2(T\mathbb{R}^n))$, $h \geq 0$, by

$$\tilde{\rho}_h(f)\xi(x, v) = \int f(x, w)\xi(x - hw, v - w) dw$$

and then obtain a more consistent representation. Note that for $h > 0$ the representations ρ_h and $\tilde{\rho}_h$ are unitarily equivalent.

(b) On a smooth Riemannian manifold M we define ρ_h by

$$\begin{aligned} (\rho_h f)\xi(x) &= \int_{T_x M} \psi(x, \exp_x(-hw)) f(x, w) \xi(\exp_x(-hw)) dw \\ &= \int_M \psi(x, y) f(x, -\exp^{-1}(x, y)/h) J_h(x, y) \xi(y) dy. \end{aligned} \quad (1.6)$$

Here $\psi \in C^\infty(M \times M)$ is a function, which is one on a neighborhood of the diagonal, $0 \leq \psi \leq 1$, and such that

$$\exp : TM \rightarrow M \times M, \quad (m, v) \mapsto (m, \exp_m v),$$

maps a neighborhood of the zero section diffeomorphically to the support of ψ , and J_h is the Jacobian determinant of the diffeomorphism

$$M \ni y \mapsto -\frac{\exp^{-1}(x, y)}{h} \in T_x M;$$

a similar construction applies to $\tilde{\rho}$.

Note that for two representations ρ_h^1, ρ_h^2 , defined with cut-off functions ψ_1 and ψ_2 , the norm $\|\rho_h^1(f) - \rho_h^2(f)\|$ tends to zero as $h \rightarrow 0$.

The following lemma is now easily established.

1.14. Lemma. To each $f \in C_c^\infty(TM)$ we associate a function $\tilde{f} \in C^\infty(TM)$ by

$$\begin{aligned} \tilde{f}(x, v, 0) &= f(x, v) \quad \text{for } h = 0, \quad x \in M, \quad v \in T_x M; \\ \tilde{f}(x, y, h) &= \psi(x, y) f(x, -\exp^{-1}(x, y)/h) \quad \text{for } h \neq 0, \quad x, y \in M. \end{aligned}$$

We then have

$$\|\varphi_h(\tilde{f})\|_{TM(h)} = \sup_x \|\pi_{(x, h)}(\tilde{f})\|_{\mathcal{L}(L^2(G_{(x, h)}, \lambda_{(x, h)}))}.$$

Moreover, since

$$\hbar^n f(x, -\exp^{-1}(x, y)/\hbar) J_\hbar(x, y) - f(x, -\exp^{-1}(x, y)/\hbar) \rightarrow 0$$

as \hbar tends to 0, we see that

$$\lim_{\hbar \rightarrow 0} \|\rho_\hbar(f) - \pi_{(x, \hbar)}(\tilde{f})\| = 0. \quad (1.7)$$

1.15. Theorem. We denote by \hat{f} the Fourier transform of f with respect to the covariable. Then

- (a) $\lim_{\hbar \rightarrow 0} \|\rho_\hbar(f)\| = \|\hat{f}\|_{\text{sup}}$;
 (b) $\lim_{\hbar \rightarrow 0} \|\rho_\hbar(f)\rho_\hbar(g) - \rho_\hbar(f * g)\| = 0$.

Proof. From (1.7) we get

$$\lim_{\hbar \rightarrow 0} \|\rho_\hbar(f)\| = \lim_{\hbar \rightarrow 0} \|\varphi_\hbar(\tilde{f})\| = \|\varphi_0(\tilde{f})\| = \|\pi_0(f)\| = \|\hat{f}\|_{\text{sup}}$$

and, for arbitrary x ,

$$\begin{aligned} \lim_{\hbar \rightarrow 0} \|\rho_\hbar(f)\rho_\hbar(g) - \rho_\hbar(f * g)\| &= \lim_{\hbar \rightarrow 0} \|\pi_{(x, \hbar)}(\tilde{f})\pi_{(x, \hbar)}(\tilde{g}) - \pi_{(x, \hbar)}(\widetilde{f * g})\| \\ &= \lim_{\hbar \rightarrow 0} \|\pi_{(x, \hbar)}(\tilde{f} * \tilde{g} - \widetilde{f * g})\| \\ &= \|\pi_0(\tilde{f} * \tilde{g} - \widetilde{f * g})\| = 0. \quad \square \end{aligned}$$

2. Manifolds with boundary

In the following, we shall denote by X a smooth n -dimensional manifold with boundary, ∂X . We assume that X is embedded in a boundaryless manifold \tilde{X} and write X° for the interior of X . We also fix a Riemannian metric on X , so that we have L^2 spaces. We will show later on that the construction is independent of the choice of the metric. First of all, however, it is helpful to study the case where $X = \mathbb{R}_+^n = \{(x_1, \dots, x_n) \mid x_n \geq 0\}$ (including $x_n = 0$!). We adopt the usual notation by writing an element $x \in \mathbb{R}_+^n$ as $x = (x', x_n)$.

Local computation of the asymptotic Green term

We change the formula for the \hbar -scaled boundary pseudodifferential operators with Fourier transformed symbol $f \in C_c^\infty(T\mathbb{R}_+^n)$ to

$$\begin{aligned} \rho_\hbar(f)\xi(x) &= \int_{x_n \geq \hbar v_n} f(x, v)\xi(x - \hbar v) dv \\ &= \hbar^{-n} \int_{w_n \geq 0} f\left(x, \frac{x - w}{\hbar}\right)\xi(w) dw, \quad \xi \in L^2(\mathbb{R}_+^n). \end{aligned} \quad (2.1)$$

A straightforward computation shows that

$$\begin{aligned} (\rho_h(f)\rho_h(g)\xi)(x) &= \int_{x_n \geq h w_n} \left(\int_{x_n \geq h v_n} f(x, v) g(x - hv, w - v) dv \right) \xi(x - hw) dw \\ &= \int_{x_n \geq h w_n} \left(\int f(x, v) g(x - hv, w - v) dv \right. \\ &\quad \left. - \int_{x_n \leq h v_n} f(x, v) g(x - hv, w - v) dv \right) \xi(x - hw) dw, \end{aligned}$$

where in the last line f, g have to be understood as extended to functions in $C_c^\infty(T\mathbb{R}^n)$. The term

$$(f *_h g)(x, w) = \int f(x, v) g(x - hv, w - v) dv \quad (2.2)$$

is just the usual composition of Fourier transformed symbols of pseudodifferential operators on manifolds without boundary. We call the remainder, i.e., the operator which maps ξ to

$$\begin{aligned} x \mapsto & - \int_{x_n \geq h w_n} \int_{x_n \leq h v_n} f(x, v) g(x - hv, w - v) dv \xi(x - hw) dw \\ &= - \int_{y_n \geq 0} \int_{x_n \leq h v_n} f(x, v) g\left(x - hv, y' - v', \frac{x_n}{h} - y_n - v_n\right) dv \xi(x' - hy', hy_n) dy \quad (2.3) \end{aligned}$$

the “asymptotic Green” term, because it corresponds to the leftover term in the composition of two truncated pseudodifferential operators in Boutet de Monvel’s calculus, which is a singular Green operator, cf. [3]. In order to analyze it, we introduce the following notation.

2.1. Definition. For $0 < h \leq 1$ define

$$\kappa_h : C_c^\infty(T\mathbb{R}^{n-1} \times \mathbb{R}_+ \times \mathbb{R}_+ \times [0, 1]) \rightarrow \mathcal{L}(L^2(\mathbb{R}_+^n))$$

by

$$\kappa_h(K)\xi(x) = \int_{y_n \geq 0} K\left(x', y', \frac{x_n}{h}, y_n, h\right) \xi(x' - hy', hy_n) dy.$$

The asymptotic Green thus is of the form $\kappa_h(l_h(f, g))$ with

$$\begin{aligned} l_h(f, g)(x', y', x_n, y_n) \\ = - \int_{x_n \leq v_n} f(x', hx_n, v) g(x' - hv', h(x_n - v_n), y' - v', x_n - y_n - v_n) dv. \quad (2.4) \end{aligned}$$

As $\hbar \rightarrow 0$ this tends to

$$l(f, g)(x', y', x_n, y_n) = - \int_{x_n \leq v_n} f(x', 0, y' - v', v_n) g(x', 0, v', x_n - v_n - y_n) dv. \quad (2.5)$$

In fact, the difference $l_{\hbar}(f, g) - l(f, g)$ is an element of $C_c^{\infty}(T\mathbb{R}^{n-1} \times \mathbb{R}_+ \times \mathbb{R}_+ \times [0, 1])$ which vanishes for $\hbar = 0$. Similarly, the difference $f *_{\hbar} g - f * g \in C_c^{\infty}(T\mathbb{R}_+^n \times [0, 1])$ vanishes for $\hbar = 0$.

In order to extend Theorem 1.15 to manifolds with boundary, the asymptotic Green term has to be taken into account.

2.2. Definition. For $\hbar = 0$ we introduce

$$\pi_0^{\partial} : C_c^{\infty}(T\mathbb{R}_+^n \times [0, 1]) \oplus C_c^{\infty}(T\mathbb{R}^{n-1} \times \mathbb{R}_+ \times \mathbb{R}_+ \times [0, 1]) \rightarrow \mathcal{L}(L^2(T\mathbb{R}^{n-1} \times \mathbb{R}_+))$$

given by

$$\begin{aligned} \pi_0^{\partial}(f \oplus K)\xi(x', v', v_n) \\ = \int_{w_n \geq 0} (f(x', 0, v' - w', v_n - w_n, 0) + K(x', v' - w', v_n, w_n, 0))\xi(x', w', w_n) dw. \end{aligned}$$

We complement π_0^{∂} by the map $\pi_0 : C_c^{\infty}(T\mathbb{R}_+^n) \rightarrow \mathcal{L}(L^2(T\mathbb{R}_+^n))$ in (1.5).

The crucial point is:

2.3. Lemma. *The map*

$$(\pi_0, \pi_0^{\partial}) : C_c^{\infty}(T\mathbb{R}_+^n) \oplus C_c^{\infty}(T\mathbb{R}^{n-1} \times \mathbb{R}_+ \times \mathbb{R}_+) \rightarrow \mathcal{L}(L^2(T\mathbb{R}_+^n) \oplus L^2(T\mathbb{R}^{n-1} \times \mathbb{R}_+))$$

given by

$$(\pi_0, \pi_0^{\partial})(f \oplus K) = (\pi_0(f), \pi_0^{\partial}(f \oplus K))$$

turns $C_c^{\infty}(T\mathbb{R}_+^n) \oplus C_c^{\infty}(T\mathbb{R}_+^{n-1} \times \mathbb{R}_+ \times \mathbb{R}_+)$ into an algebra. We denote this product with $*'$. Note that $f *' g = f * g + l(f, g)$.

It is clear that Theorem 1.15(b) will not remain true literally. Instead we obtain:

2.4. Theorem. *For two symbols $f, g \in C_c^{\infty}(T\mathbb{R}_+^n)$ the following holds:*

$$\lim_{\hbar \rightarrow 0} \|\rho_{\hbar}(f)\rho_{\hbar}(g) - \rho_{\hbar}(f * g) - \kappa_{\hbar}(l(f, g))\| = 0.$$

As in the case of manifolds without boundary, this will be related to the continuity of a field of C^* -algebras which we will now introduce.

2.5. Definition. We denote by A the C^* -completion of

$$A^\infty = C_c^\infty(T\mathbb{R}_+^n \times [0, 1]) \oplus C_c^\infty(T\mathbb{R}^{n-1} \times \mathbb{R}_+ \times \mathbb{R}_+ \times [0, 1])$$

in the representation $\rho_h + \kappa_h$, for $h \neq 0$ and (π_0, π_0^∂) for $h = 0$, i.e., under the mappings

$$f \oplus K \mapsto \begin{cases} \rho_h(f) + \kappa_h(K), & h \neq 0, \\ \pi_0(f) \oplus \pi_0^\partial(f \oplus K), & h = 0. \end{cases}$$

There are obvious maps

$$\varphi_h : A \rightarrow A(h),$$

where $A(h)$ is the completion of $C_c^\infty(T\mathbb{R}_+^n) \oplus C_c^\infty(T\mathbb{R}^{n-1} \times \mathbb{R}_+ \times \mathbb{R}_+)$ with respect to the specific representation in h .

We will show:

2.6. Theorem. *The triple $(A, \{A(h), \varphi_h\}_{h \in [0, 1]})$ is a continuous field of C^* -algebras with $A(h)$ isomorphic to the compact operators for $h \neq 0$.*

For fixed $h \neq 0$, the operators $\rho_h(f) + \kappa_h(K)$ are compact, because they are integral operators with a square integrable kernel, so $A(h)$ is isomorphic to the compact operators.

We shall next analyze the field in more detail. We abbreviate

$$T = T\mathbb{R}_+^n \quad \text{and} \quad T_\partial = T\mathbb{R}^{n-1} \times \mathbb{R}_+ \times \mathbb{R}_+$$

and start with the following observation.

2.7. Proposition. *As a subset of A , A^∞ is a $*$ -algebra.*

Proof. First we prove closure under multiplication. The product of $K_1, K_2 \in C_c^\infty(T_\partial \times [0, 1])$ is just the convolution product of the two functions on the groupoid $T\mathbb{R}^{n-1} \times \mathbb{R}_+ \times \mathbb{R}_+$, where $\mathbb{R}_+ \times \mathbb{R}_+$ carries the pair groupoid structure. Thus it is again a function in $C_c^\infty(T_\partial \times [0, 1])$.

For $f, g \in C_c^\infty(T)$ we have already computed, cf. (2.2), (2.4):

$$\rho_h(f)\rho_h(g) = \rho_h(\tilde{f} *_h \tilde{g}) + \kappa_h(l_h(\tilde{f}, \tilde{g})),$$

where \tilde{f}, \tilde{g} are smooth extensions of f, g to functions in $C_c^\infty(T\mathbb{R}^n \times [0, 1])$. Since

$$(\pi_0, \pi_0^\partial)(f)(\pi_0, \pi_0^\partial)(g) = (\pi_0(f * g), \pi_0^\partial(f * g) + \pi_0^\partial(l(f, g)))$$

we see the closure under products of f, g .

Checking the closure under products of f 's with K 's is straightforward. The same is true for the closure under involution. \square

The algebra in 0

The algebra in zero, $A(0)$, is the completion of

$$A(0)^\infty := (C_c^\infty(T) \oplus C_c^\infty(\mathcal{T}_\partial), *)'$$

in the representation (π_0, π_0^∂) . The summand $C_c^\infty(\mathcal{T}_\partial)$ becomes an ideal in $A(0)^\infty$. We thus get the short exact sequence

$$0 \rightarrow C_c^\infty(\mathcal{T}_\partial) \rightarrow A(0)^\infty \xrightarrow{q} C_c^\infty(T) \rightarrow 0. \quad (2.6)$$

As noted in the proof of Proposition 2.7, the algebra structure on $C_c^\infty(\mathcal{T}_\partial)$ comes from the groupoid structure on \mathcal{T}_∂ . Likewise, the algebra structure on $C_c^\infty(T)$ stems from the groupoid structure on T . Note that both groupoids are amenable.

2.8. Lemma. *We have a short exact sequence of C^* -algebras*

$$0 \rightarrow C_r^*(\mathcal{T}_\partial) \rightarrow A(0) \rightarrow C_r^*(T) \rightarrow 0. \quad (2.7)$$

Proof. In the short exact sequence (2.6), the projection q , mapping $f \oplus K$ to f , is a $*$ -homomorphism. The trivial estimate

$$\|\pi_0(f)\|_{\mathcal{L}(L^2(T))} \leq \|\pi_0(f) \oplus \pi_0^\partial(f \oplus K)\|_{\mathcal{L}(L^2(T) \oplus L^2(T\mathbb{R}^{n-1} \times \mathbb{R}_+))},$$

shows that π extends to a map $A(0) \rightarrow C_r^*(T)$ with $C_r^*(\mathcal{T}_\partial)$ in its kernel. Since we may estimate the norm of $\pi_0^\partial(f)$ by the norm of $\pi_0(f)$, we obtain (2.7). \square

Alternatively, the lemma may be proven using only the amenability of the groupoids, similarly as in the proof of Theorem 2.11, below. Note that, via the Fourier transform,

$$C_r^*(\mathcal{T}_\partial) \simeq C_0(T^*\mathbb{R}^{n-1}) \otimes \mathcal{K}(L^2(\mathbb{R}_+))$$

and

$$C_r^*(T) \simeq C_0(T^*\mathbb{R}_+^n).$$

Upper semi-continuity

2.9. Definition. On A we define

$$\|a\|_{as} = \max\left(\limsup_{h \rightarrow 0} \|\varphi_h(a)\|, \|\varphi_0(a)\|\right).$$

This is a C^* -seminorm which is continuous with respect to the norm of A . The quotient

$$A[0] = A/I, \quad \text{where } I = \{a \in A \mid \|a\|_{as} = 0\},$$

therefore carries two norms: the quotient norm and $\|\cdot\|_{as}$. Both are equivalent by [7, Proposition 1.8.1], so that $A[0]$ is a C^* -algebra with norm $\|\cdot\|_{as}$.

Since $\|a\|_{as} \geq \|\varphi_0(a)\|$ we have a natural map

$$\Phi : A[0] \rightarrow A(0).$$

2.10. Lemma. *Elements in A^∞ which are 0 for $\hbar = 0$ belong to I .*

Proof. For $f \oplus K \in A^\infty$ it is easy to estimate

$$\|\rho_\hbar(f)\| \leq M_f \|f(\cdot, \hbar)\|_\infty \quad \text{and} \quad \|\kappa_\hbar(K)\| \leq M_K \|K(\cdot, \hbar)\|_\infty,$$

where M_f and M_K are constants depending on f and K , respectively, but not on \hbar . \square

2.11. Theorem. *The field $(A, \{A(\hbar), \varphi_\hbar\}_{\hbar \in [0,1]})$ is upper semi-continuous in 0.*

Proof. We denote by R the closure of the range of the natural map $\gamma : C_c^\infty(\mathcal{T}_\partial) \rightarrow A[0]$. This is an ideal in $A[0]$. Indeed, $C_c^\infty(\mathcal{T}_\partial)$ is an ideal in $A(0)^\infty$, and the extension (e.g. constant in \hbar) of functions in $A(0)^\infty$ to functions in A^∞ furnishes an embedding of $A(0)^\infty$ into $A[0]$ with dense range.

Since \mathcal{T}_∂ is amenable, the quotient map $C^*(\mathcal{T}_\partial) \rightarrow C_r^*(\mathcal{T}_\partial)$ is an isomorphism. It factorizes through R , since R gives us a Hilbert space representation of \mathcal{T}_∂ , while $\|a\|_{as} \geq \|\varphi_0(a)\|$.

This leads to a commutative diagram of natural maps

$$\begin{array}{ccc} & C^*(\mathcal{T}_\partial) & \\ \nearrow & \downarrow & \\ C_c^\infty(\mathcal{T}_\partial) \hookrightarrow & R \subseteq A[0] & \\ \searrow & \downarrow & \\ & C_r^*(\mathcal{T}_\partial) & \end{array}$$

where the upper vertical arrow is surjective, since the inclusion has dense range. The invertibility of the quotient map implies that the lower vertical arrow is an isomorphism.

Next we define a map $\tilde{q} : A[0] \rightarrow C_r^*(T)$. By definition, $A[0]$ is the set of equivalence classes of Cauchy sequences in A^∞ with respect to $\|\cdot\|_{as}$. Given such a Cauchy sequence $a_k = (f_k \oplus K_k)$, we may evaluate at $\hbar = 0$ and obtain a sequence $(f_k^0 \oplus K_k^0)$ in $A(0)^\infty$. As $\|a_k\|_{as} \geq \|\varphi_0(a_k)\|$, the sequence (f_k^0) is a Cauchy sequence in $C_r^*(T)$; moreover, the mapping $(a_k) \mapsto (f_k^0)$ is well defined and continuous. In view of Lemma 2.10 its kernel is R .

Combining this with the short exact sequence (2.7) we obtain the following commutative diagram of short exact sequences:

$$\begin{array}{ccccccc} 0 & \rightarrow & C_r^*(\mathcal{T}_\partial) & \rightarrow & A[0] & \xrightarrow{\tilde{q}} & C_r^*(T) \rightarrow 0 \\ \parallel & & \parallel & & \downarrow \Phi & & \parallel \\ 0 & \rightarrow & C_r^*(\mathcal{T}_\partial) & \rightarrow & A(0) & \rightarrow & C_r^*(T) \rightarrow 0 \end{array} \quad (2.8)$$

We conclude from the five lemma that Φ is an isomorphism, and therefore

$$\limsup_{\hbar \rightarrow 0} \|\varphi_\hbar(a)\| \leq \|\varphi_0(a)\|,$$

i.e., the field is upper semi-continuous in 0. \square

What is still missing is the proof of the lower semi-continuity of the field A . It will be given at the end of Section 2, since there is no simplification for the half-space case.

The tangent groupoid for a manifold with boundary

2.12. Definition. We denote by T^-X the subset of $T\tilde{X}$ formed by all vectors $(m, v) \in T\tilde{X}|_X$ for which $\exp_m(-\varepsilon v) \in X$ for sufficiently small $\varepsilon > 0$. This is a semi-groupoid with addition of vectors. Note that $T^-X = TX^\circ \cup T^-X|_{\partial X}$.

We define T^-X as the disjoint union $T^-X \cup (X \times X \times]0, 1])$, endowed with the fiberwise semi-groupoid structure induced by the semi-groupoid structure on T^-X and the groupoid structure on $X \times X$. As in the boundaryless case, we glue T^-X to $X \times X \times]0, 1]$ via the charts

$$T^-X \times [0, 1] \supseteq U \ni (m, v, \hbar) \mapsto \begin{cases} (m, v) & \text{for } \hbar = 0, \\ (m, \exp_m(-\hbar v), \hbar) & \text{for } \hbar \neq 0 \end{cases}$$

and let $T^-X(0) = T^-X$ and $T^-X(\hbar) = X \times X \times \{\hbar\}$.

In order to avoid problems with the topology of T^-X (which is in general not a manifold with corners) we let $C_c^\infty(T^-X) = C_c^\infty(T\tilde{X})|_{T^-X}$.

C^ -algebras associated to the semi-groupoids T^-X and T^-X*

We start with T^-X . Let $C_c^\infty(T^-X)$ denote the smooth functions on T^-X which have compact support in T^-X . In analogy with Definition 2.1 we introduce

$$\begin{aligned} \pi_0 : C_c^\infty(T^-X) &\rightarrow \mathcal{L}(L^2(TX^\circ)) \quad \text{and} \\ \pi_0^\partial : C_c^\infty(T^-X) &\rightarrow \mathcal{L}(L^2(T^+X|_{\partial X})) \end{aligned}$$

acting by

$$\pi_0(f)\xi(m, v) = \int_{T_m X} f(m, v - w)\xi(m, w)dw, \quad (2.9)$$

$$\pi_0^\partial(f)\xi(m, v) = \int_{T_m^+ X} f(m, v - w)\xi(m, w)dw. \quad (2.10)$$

Note that due to its compact support in T^-X , the function f naturally extends (by zero) to TX .

2.13. Definition. We denote by $C_r^*(T^-X)$ the C^* -algebra generated by π_0 and π_0^∂ , i.e., by the map $C_c^\infty(T^-X) \ni f \mapsto (\pi_0(f), \pi_0^\partial(f)) \in \mathcal{L}(L^2(TX^\circ) \oplus L^2(T^+X|_{\partial X}))$.

At first glance, this definition seems to overlook the operators of the form $\pi_0^\partial(K)$ in 2.1 and operators of the form $\pi_0(f)$ and $\pi_0^\partial(f)$, where $f \in C_c^\infty(TX) = C_c^\infty(T\tilde{X})|_{TX}$. In fact, this is not the case. The second type of operators belongs to $C_r^*(T^-X)$, because we take the closure under the adjoint operation and addition. The reason that the first type of operators is in $C_r^*(T^-X)$, is the well-known relation between operators of half-convolution and Toeplitz operators, which we

recall, below. We denote by \mathfrak{T} the algebra of all Toeplitz operators on $L^2(S^1)$ and by \mathfrak{T}_0 the ideal of all operators whose symbol vanishes in -1 .

2.14. Lemma. *Let $f \in C_c^\infty(\mathbb{R})$. Then the operator*

$$L^2(\mathbb{R}_+) \ni \xi \mapsto \left(s \mapsto \int_0^\infty f(s-w)\xi(w)dw \right) \in L^2(\mathbb{R}_+)$$

is unitarily equivalent to the Toeplitz operator T_φ with symbol $\varphi(z) = \hat{f}(i(z-1)/(z+1))$. Note that $\varphi(-1) = \hat{f}(\infty) = 0$.

The C^ -algebra generated by the operators in the image of $C_c^\infty(\mathbb{R})$ under this map is precisely the ideal \mathfrak{T}_0 , while the compact operators in \mathfrak{T} are generated by their commutators.*

Proof. Plancherel's theorem shows that the above operator of half-convolution is the truncated pseudodifferential operator with symbol \hat{f} , mapping $\xi \in L^2(\mathbb{R}_+)$ to $\text{op}(\hat{f})_+\xi(s) = \int e^{ist} \hat{f}(t) \widehat{(e^+\xi)}(t) dt$, where $e^+\xi$ is the extension (by zero) of ξ to \mathbb{R} .

Now one observes that the unitary $U : L^2(S^1) \rightarrow L^2(\mathbb{R})$ given by

$$Ug(t) = \frac{\sqrt{2}}{1+it} g\left(\frac{1-it}{1+it}\right)$$

maps the Hardy space H^2 to $F(L^2(\mathbb{R}_+))$ with the Fourier transform F , and that $\text{op}(\hat{f})_+$ is $F^{-1}UT_\varphi U^{-1}F$. See [18, Section 2] for details.

For the second statement, one first notes that the C^* -algebra generated by these operators is a subalgebra of \mathfrak{T}_0 . On the other hand, \mathfrak{T}_0 consists of the operators of the form $T_\varphi + C$, where $\varphi \in C(S^1)$ vanishes in -1 , and C is compact. According to [8, Proposition 7.12], the commutators of all T_φ , $\varphi \in C(S^1)$, generate the compacts, hence so do the commutators of those T_φ , where φ vanishes in -1 . As these T_φ can be approximated by elements in the image of $C_c^\infty(\mathbb{R})$, the proof is complete. \square

2.15. Lemma. *We have a representation π_0^∂ of $C_c^\infty(T\partial X \times \mathbb{R}_+ \times \mathbb{R}_+)$ on $L^2(T^+X|_{\partial X})$ via*

$$\pi_0^\partial(K)\xi(m, v', v_n) = \int_{T^+X|_{\partial X}} K(m, v' - w', v_n, w_n)\xi(m, w', w_n)dw'dw_n. \quad (2.11)$$

Through the isomorphism $C_r^(T\partial X) \xrightarrow{\sim} C_0(T^*\partial X)$ given by the Fourier transform, the closure of the range of π_0^∂ is isomorphic to*

$$J = C_0(T^*\partial X) \otimes \mathcal{K}(L^2(\mathbb{R}_+)).$$

J is (isomorphic to) an ideal in $C_r^(T^-X)$ generated by commutators of elements of the form $\pi_0^\partial(f)$.*

Proof. The algebraic tensor product $C_c^\infty(T\partial X) \otimes C_c^\infty(\mathbb{R}_+ \times \mathbb{R}_+)$ is dense in $C_c^\infty(T\partial X \times \mathbb{R}_+ \times \mathbb{R}_+)$. Due to the continuity of

$$\pi_0^\partial : C_c^\infty(T\partial X \times \mathbb{R}_+ \times \mathbb{R}_+) \rightarrow \mathcal{L}(L^2(T^+X|_{\partial X}))$$

it is sufficient to determine the closure of $\pi_0^\partial(C_c^\infty(T\partial X) \otimes C_c^\infty(\mathbb{R}_+ \times \mathbb{R}_+))$.

It is clear that $\pi_0^\partial(C_c^\infty(T\partial X) \otimes C_c^\infty(\mathbb{R}_+ \times \mathbb{R}_+)) \subseteq J$. In fact, we have equality, since $C_r^*(T\partial X) \xrightarrow{\sim} C_0(T^*\partial X)$ and since a compact operator on $L^2(\mathbb{R}_+)$ can be approximated by a Hilbert–Schmidt operator, thus by an integral operator with kernel in $C_c^\infty(\mathbb{R}_+ \times \mathbb{R}_+)$.

In order to see that J is contained in $C_r^*(T^-X)$, it is sufficient to approximate both factors of a pure tensor $h \otimes c$, where $h \in C_0(T^*\partial X)$ and $c \in \mathcal{K}(L^2(\mathbb{R}_+))$. For the first task we choose a function in $C_c^\infty(T\partial X)$ whose fiberwise Fourier transform is close to h in sup-norm. For the second, we refer to Lemma 2.14. In particular, we see that J also is generated by commutators.

A direct computation shows that J is an ideal in $C_r^*(T^-X)$. \square

2.16. Definition. We let

$$C_{ic}^\infty(T^-X) = C_c^\infty(TX) \oplus C_c^\infty(T\partial X \times \mathbb{R}_+ \times \mathbb{R}_+).$$

This is a dense $*$ -subalgebra of $C_r^*(T^-X)$. We will denote the product in this subalgebra by $*'$.

For $\hbar \neq 0$ we obtain representations of $C_c^\infty(T^-X) = C_c^\infty(\mathcal{T}\tilde{X})|_{T^-X}$ in $\mathcal{L}(L^2(X))$ by

$$\pi_\hbar(f)\xi(m) = \frac{1}{\hbar^n} \int f(m, \tilde{m}, \hbar)\xi(\tilde{m})d\tilde{m}. \quad (2.12)$$

Note that these are the natural groupoid representations for $X \times X \times]0, 1]$.

We denote by $C_r^*(T^-X)$ the reduced C^* -algebra generated by π_\hbar , $0 \leq \hbar \leq 1$, and π_0^∂ .

For $X = \mathbb{R}_+^n$ we have $C_{ic}^\infty(T^-X) = A(0)^\infty$, $C_r^*(T^-X) = A(0)$ and $C_r^*(\mathcal{T}^-X) = A$. Also there are evaluation maps

$$\varphi_\hbar : C_r^*(\mathcal{T}^-X) \rightarrow C_r^*(T^-X)(\hbar).$$

We can now formulate the main result of this section:

2.17. Theorem. *We have*

$$\begin{aligned} C_r^*(\mathcal{T}^-X(\hbar)) &= \mathcal{K}(L^2(X)), \quad \hbar \neq 0, \\ C_r^*(\mathcal{T}^-X(0)) &= C_r^*(T^-X). \end{aligned}$$

Moreover, $(C_r^*(T^-X), \{C_r^*(\mathcal{T}^-X(\hbar)), \varphi_\hbar\}_{\hbar \in [0,1]})$ is a continuous field of C^* -algebras.

The first two statements are by now obvious: on one hand, operators with smooth integral kernels are compact; on the other, we analyzed the algebra in 0 in 2.13 and 2.15.

In order to show continuity, we shall establish first upper and then lower semi-continuity. For the proof of upper semi-continuity, we will essentially follow the ideas for the half-space case. Our first task is the construction of a representation of $C_{ic}^\infty(T^-X)$. To this end, we will

simply extend $f \in C_c^\infty(TX)$ and $K \in C_c^\infty(T\partial X \times \mathbb{R}_+ \times \mathbb{R}_+)$ to functions \tilde{f} and \tilde{K} on T^-X as described below, then apply (2.12).

Choose a function $\psi \in C^\infty(X \times X)$ which is one on a neighborhood of the diagonal, $0 \leq \psi \leq 1$, such that

$$\exp: T^-X \rightarrow X \times X$$

maps a neighborhood of the zero section diffeomorphically to the support of ψ .

For $f \in C_c^\infty(TX)$ we define $\tilde{f} \in C_c^\infty(T^-X)$ by

$$\tilde{f}(m, \tilde{m}, \hbar) = \psi(m, \tilde{m}) f\left(m, -\frac{\exp^{-1}(m, \tilde{m})}{\hbar}\right). \quad (2.13)$$

We next identify a neighborhood U of ∂X in X with $\partial X \times [0, 1[$ and write $U \ni m = (m', m_n)$ with $m' \in \partial X$ and $m_n \geq 0$. We also choose a function $\chi \in C_c^\infty(X)$ supported in U with $0 \leq \chi \leq 1$ and $\chi \equiv 1$ near ∂X . For $K \in C_c^\infty(T\partial X \times \mathbb{R}_+ \times \mathbb{R}_+)$ we then define $\tilde{K} \in C_c^\infty(X \times X \times]0, 1])$ by

$$\tilde{K}(m, \tilde{m}, \hbar) = \chi(m)\chi(\tilde{m})\psi(m, \tilde{m})K\left(m', -\frac{\exp^{-1}(m', \tilde{m}')}{\hbar}, \frac{m_n}{\hbar}, \frac{\tilde{m}_n}{\hbar}\right). \quad (2.14)$$

2.18. Remark. In the half-space case with the flat metric we have, for fixed f and K ,

$$\pi_\hbar(\tilde{f}) = \rho_\hbar(f) \quad \text{and} \quad \pi_\hbar(\tilde{K}) = \kappa_\hbar(K)$$

provided \hbar is sufficiently small.

2.19. Corollary. From Theorem 2.17 we then obtain the analog of property (0.1):

$$\lim_{\hbar \rightarrow 0} \|\pi_\hbar(\tilde{f}) + \pi_\hbar(\tilde{K})\| = \max\{\|\pi_0(f)\|, \|\pi_0^\partial(f \oplus K)\|\}. \quad (2.15)$$

2.20. Metrics. The construction of $C_r^*(T^-X)$ and the extensions (2.13), (2.14) used a metric, but $C_r^*(T^-X)$ is independent of the choice. Let ν_1, ν_2 be two different metrics on X , and denote by μ_1, μ_2 the associated measures on X as well as the fiberwise measures in TX . Let $k \in C^\infty(X)$ be given by

$$\mu_1 = k\mu_2.$$

Multiplication by \sqrt{k} yields a unitary

$$U: (L^2(X), \mu_1) \rightarrow (L^2(X), \mu_2),$$

and multiplication by $\sqrt{k(m)}$ a family of unitaries

$$U_m: (L^2(T_m^-X), \mu_1) \rightarrow (L^2(T_m^-X), \mu_2).$$

We define

$$\phi: C_c^\infty(T^-X) \rightarrow C_c^\infty(T^-X)$$

taking $f(m, v, 0)$ to $f(m, v, 0)k(m)$ for $\hbar = 0$ and $f(m, \tilde{m}, \hbar)$ to $f(m, \tilde{m}, \hbar)\sqrt{k(m)k(\tilde{m})}$, $\hbar \neq 0$. Then $\pi_{\hbar}^1(f) = U^{-1}\pi_{\hbar}^2(\phi(f))U$, where π_{\hbar}^1 and π_{\hbar}^2 are the representations induced by μ_1 and μ_2 . A corresponding relation holds for π_0^{∂} . Hence $C_r^*(T^-X)$ is independent of the metric.

The following lemma clarifies the influence of the extension by different metrics.

2.21. Lemma. *Let $f \in C_c^\infty(TX)$. Denote by \tilde{f}^i the extension of f with respect to the metric v_i , $i = 1, 2$. Then*

$$\|\pi_{\hbar}(\phi(\tilde{f}^1)) - \pi_{\hbar}(\widetilde{\phi(f)^2})\| \rightarrow 0 \quad \text{for } \hbar \rightarrow 0.$$

Here π_{\hbar} is understood with respect to μ_2 .

Proof. This follows from Lemma 2.10, since $\phi(\tilde{f}^1) - \widetilde{\phi(f)^2}$ is a function in $C_c^\infty(T^-X)$ which is zero at $\hbar = 0$. \square

A similar statement holds if we start with $K \in C_c^\infty(T\partial X \times \mathbb{R}_+ \times \mathbb{R}_+)$.

Upper semi-continuity

We again use the seminorm

$$\|a\|_{as} = \max\left\{\|\varphi_0(a)\|, \limsup_{\hbar \rightarrow 0}\|\varphi_{\hbar}(a)\|\right\}$$

for elements in $C_r^*(T^-X)$ and introduce the analog of $A[0]$:

$$C_{\gamma}^*(T^-X) = C_r^*(T^-X)/I,$$

where

$$I = \{a \in C_r^*(T^-X) \mid \|a\|_{as} = 0\}.$$

The notation $C_{\gamma}^*(T^-X)$ is justified by the following:

2.22. Proposition. *The mappings $f \mapsto \tilde{f}$ and $K \mapsto \tilde{K}$ induce a $*$ -homomorphism Ψ from $(C_{ic}^\infty(T^-X), *)$ to $C_{\gamma}^*(T^-X)$ with dense range. We have*

$$\lim_{\hbar \rightarrow 0} \|\pi_{\hbar}(\tilde{f})\pi_{\hbar}(\tilde{g}) - \pi_{\hbar}(\widetilde{f *' g})\| = 0, \quad f, g \in C_c^\infty(TX). \quad (2.16)$$

Proof. Choose an open covering $\{U_i\}$ of X , where each U_i can be identified with an open subset of \mathbb{R}^n or \mathbb{R}_+^n . By possibly shrinking the U_i , we may assume that the function ψ used in (2.13) and (2.14) equals 1 on $U_i \times U_i$ and that the function χ is $\equiv 1$ on U_i whenever U_i intersects the boundary. We also fix a subordinate partition of unity $\{\psi_i\} \subset C_c^\infty(U_i)$.

For $f, g \in C_c^\infty(TX)$ we have $(\psi_i f) *' g = (\psi_i f) *' (\eta_i g)$ for each $\eta_i \in C_c^\infty(U_i)$ with $\psi_i \eta_i = \psi_i$. Moreover, $\pi_{\hbar}(\psi_i f)\pi_{\hbar}(\tilde{g}) = \pi_{\hbar}(\psi_i f)\pi_{\hbar}(\theta_i g)$ for suitable $\theta_i \in C_c^\infty(U_i)$, provided \hbar is small. Hence

$$\begin{aligned} \|\pi_h(\widetilde{f *' g}) - \pi_h(\widetilde{f})\pi_h(\widetilde{g})\| &\leq \sum \|\pi_h((\widetilde{\psi_i f}) *' g) - \pi_h(\widetilde{\psi_i f})\pi_h(\widetilde{g})\| \\ &= \sum \|\pi_h((\widetilde{\psi_i f}) *' \eta_i g) - \pi_h(\widetilde{\psi_i f})\pi_h(\widetilde{\theta_i g})\|. \end{aligned} \quad (2.17)$$

For sufficiently small \hbar , all operators will have support in $U_i \times U_i \times [0, 1]$ so that we are working on Euclidean space. According to Lemma 2.21 we can also, modulo terms converging to zero as $\hbar \rightarrow 0$, use the Euclidean metric. So we are precisely in the situation considered at the beginning of the section. The explicit computation shows that

$$\pi_h(\widetilde{f *' g}) - \pi_h(\widetilde{f})\pi_h(\widetilde{g}) = \rho_h(f * g - f *_{\hbar} g) + \kappa_h(l(f, g) - l_h(f, g)). \quad (2.18)$$

As $f * g - f *_{\hbar} g \in C_c^\infty(T\mathbb{R}_+^n \times [0, 1])$ and $l(f, g) - l_h(f, g) \in C^\infty(T\mathbb{R}_+^{n-1} \times \mathbb{R}_+ \times \mathbb{R}_+ \times [0, 1])$ vanish for $\hbar = 0$, the difference (2.18) is in I by Lemma 2.10. Hence (2.17) tends to zero, and $\Psi(f *' g) = \Psi(f)\Psi(g)$. The remaining $*$ -algebra properties are checked similarly.

In order to see that the image of Ψ is dense in $C_\gamma^*(T^-X)$, we simply note that the evaluation at $\hbar = 0$ associates to an element F in $C_c^\infty(T^-X)$ an element in $C_{lc}^\infty(T^-X)$ whose extension via (2.13), (2.14) induces the same element in $C_\gamma^*(T^-X)$ by Lemma 2.10. \square

2.23. Remark. Property (2.16) is the analog of the asymptotic multiplicativity (0.2) in the case of manifolds with boundary. In particular, we have established Theorem 2.4.

With Proposition 2.22, the proof of the following theorem is analogous to that of Theorem 2.11.

2.24. Theorem. $(C_r^*(T^-X), \{C_r^*(T^-X)(\hbar), \varphi_\hbar\}_{\hbar \in [0, 1]})$ is upper semi-continuous in 0.

Lower semi-continuity

As in the classical case [10] lower semi-continuity is proven by introducing strongly continuous representations using the groupoid structure. We split the representations into two: one taking care of the contribution from the interior of the manifold, i.e., the convolution part, and one taking care of the boundary part, i.e., half-convolution and kernels on the boundary.

For the lemmas, below, we note that—by construction— π_0 and π_0^∂ extend to $C_r^*(T^-X)$.

2.25. Lemma. $\liminf_{\hbar \rightarrow 0} \|\varphi_\hbar(a)\| \geq \|\pi_0(a)\|$ for all $a \in C_r^*(T^-X)$.

Proof. According to Proposition 2.22 it is sufficient to show that

$$\lim_{\hbar \rightarrow 0} \|\pi_h(\widetilde{f} + \widetilde{K})\| \geq \|\pi_0(f)\| \quad \text{for } f \oplus K \in C_{lc}^\infty(T^-X). \quad (2.19)$$

For $g \in C_c^\infty(T^-X)$ define

$$\|g\|_{\infty, \hbar}^2 = \sup_{m \in X} \left\{ \frac{1}{\hbar^n} \int_X |g(x, m, \hbar)|^2 dx \right\} \quad \text{for } \hbar \neq 0,$$

and

$$\|g\|_{\infty,0}^2 = \sup_{m \in X} \left\{ \int_{T_m X} |g(m, v, 0)|^2 dv \right\} \quad \text{for } \hbar = 0.$$

Set

$$\|g\|_{\infty} = \sup_{\hbar \in [0,1]} \|g\|_{\infty,\hbar}.$$

It is easily checked that

$$\|\pi_{\hbar}(\tilde{f} + \tilde{K})\| = \sup \left\{ \left\| \frac{1}{\hbar^n} \int (\tilde{f}(\cdot, m, \hbar) + \tilde{K}(\cdot, m, \hbar)) g(m, \cdot, \hbar) dm \right\|_{\infty,\hbar} \mid \|g\|_{\infty} \leq 1 \right\} \quad (2.20)$$

for $\hbar \neq 0$, and

$$\|\pi_0(f)\| = \sup \left\{ \left\| \int f(\cdot, v, 0) g(\cdot, \cdot - v, 0) dv \right\|_{\infty,0} \mid \|g\|_{\infty} \leq 1 \right\}. \quad (2.21)$$

In fact, for (2.20) we note that “ \geq ” follows from the estimate

$$\begin{aligned} & \left\| \frac{1}{\hbar^n} \int \tilde{f}(m_1, m, \hbar) g(m, m_2, \hbar) dm \right\|_{\infty,\hbar}^2 \\ &= \left\| \frac{1}{\hbar^n} \pi_{\hbar}(\tilde{f}) g(\cdot, m_2, \hbar) \right\|_{\infty,\hbar}^2 \\ &= \sup_{m_2 \in X} \frac{1}{\hbar^n} \|\pi_{\hbar}(\tilde{f}) g(\cdot, m_2, \hbar)\|_{L^2(X)}^2 \leq \|\pi_{\hbar}(\tilde{f})\|^2 \sup_{m_2 \in X} \frac{1}{\hbar^n} \|g(\cdot, m_2, \hbar)\|_{L^2(X)}^2 \\ &= \|\pi_{\hbar}(\tilde{f})\|^2 \|g\|_{\infty,\hbar}^2 \leq \|\pi_{\hbar}(\tilde{f})\|^2 \|g\|_{\infty}^2. \end{aligned}$$

For the reverse inequality we choose $g(x, m, \hbar) = s(m) \xi(x) \hbar^n \varphi(\hbar)$, where $s \in C_c^\infty(X)$, $s \leq 1$, $\|\xi\|_{L^2(X)} = 1$ with $\|\pi_{\hbar}(\tilde{f}) \xi\| \geq \|\pi_{\hbar}(\tilde{f})\| - \varepsilon$, and $\varphi \in C_c^\infty([0, 1])$ is equal to one outside a neighborhood of zero. Equation (2.21) follows by a similar argument.

Now suppose that $g \in C_c^\infty(\mathcal{T}^-X)$ and $g(x, m, \hbar) = 0$ for $x \in \partial X$. Then the weak convergence of \tilde{K} towards zero implies that

$$\lim_{\hbar \rightarrow 0} \left\| \frac{1}{\hbar^n} \int (\tilde{f}(\cdot, m, \hbar) + \tilde{K}(\cdot, m, \hbar)) g(m, \cdot, \hbar) dm \right\|_{\infty,\hbar} = \left\| \int f(\cdot, v, 0) g(\cdot, \cdot - v, 0) dv \right\|_{\infty,0}.$$

As the set of these g is dense in $\{g \in C_c^\infty(\mathcal{T}^-X) \mid \|g\|_{\infty} \leq 1\}$, (2.19) follows. \square

2.26. Lemma. $\liminf_{\hbar \rightarrow 0} \|\varphi_{\hbar}(a)\| \geq \|\pi_0^{\partial}(a)\|$ for all $a \in C_r^*(\mathcal{T}^-X)$.

Proof. As in the proof of Lemma 2.25 we only have to show that

$$\liminf_{\hbar \rightarrow 0} \|\pi_{\hbar}(\tilde{f} + \tilde{K})\| \geq \|\pi_0^{\partial}(f \oplus K)\|, \quad (2.22)$$

for $f \in C_c^{\infty}(TX)$ and $K \in C_c^{\infty}(T\partial X \times \mathbb{R}_+ \times \mathbb{R}_+)$.

We let P_{\hbar} be the projection in $L^2(X)$ given by multiplication by the characteristic function of $\partial X \times [0, a_{\hbar}[$, where

$$a_{\hbar} \rightarrow 0 \quad \text{for } \hbar \rightarrow 0 \quad \text{and} \quad \frac{a_{\hbar}}{\hbar} \rightarrow \infty \quad \text{for } \hbar \rightarrow 0.$$

As $\|P_{\hbar}\pi_{\hbar}(\tilde{f} + \tilde{K})P_{\hbar}\| \leq \|\pi_{\hbar}(\tilde{f} + \tilde{K})\|$, it is enough to show that

$$\liminf_{\hbar \rightarrow 0} \|P_{\hbar}\pi_{\hbar}(\tilde{f} + \tilde{K})P_{\hbar}\| \geq \|\pi_0^{\partial}(f \oplus K)\|.$$

Since we are free to choose a metric, we fix a metric on ∂X and the standard metric on $[0, a_{\hbar}[$.

As in the proof of Lemma 2.25, we equip the space $C_c^{\infty}(T\partial X \times [0, \infty[)$ with norms $\|\cdot\|_{\infty, \hbar}$, $\|\cdot\|_{\infty}$, which are just like the norms before, on $T\partial X$ instead of T^-X , combined with the L^2 -norm on $[0, \infty[$. For $f \in C_c^{\infty}(T^-X)$ and $K \in C_c^{\infty}(T\partial X \times \mathbb{R}_+ \times \mathbb{R}_+)$ we define representations on $C_c^{\infty}(T\partial X \times [0, \infty[)$ by

$$\eta_{\hbar}(f)g(m_1, m_2, \hbar, b) = \frac{1}{\hbar^{n-1}} \int_{a \in [0, a_{\hbar}/\hbar]} f(m_1, \hbar b, m, \hbar a, \hbar)g(m, m_2, \hbar, a) dm da,$$

$$b \in \left[0, \frac{a_{\hbar}}{\hbar}\right], \quad \hbar \neq 0,$$

$$\eta_0(f)g(m_1, v, 0, b) = \int_{T_{m_1}\partial X \times \mathbb{R}_+} f(m_1, 0, v - w, b - a, 0)g(m_1, w, 0, a) dw da,$$

$$\eta_0(K)g(m_1, v, 0, b) = \int_{T_{m_1}\partial X \times \mathbb{R}_+} K(m_1, v - w, b, a)g(m_1, w, 0, a) dw da.$$

Note that $\|P_{\hbar}\pi_{\hbar}(f)P_{\hbar}\| = \|D_{\hbar}P_{\hbar}\pi_{\hbar}(f)P_{\hbar}D_{\hbar^{-1}}\| = \sup\{\|\eta_{\hbar}(f)g\|_{\infty, \hbar} \mid \|g\|_{\infty} \leq 1\}$, where D_{\hbar} is the dilation operator in the normal direction, given by $D_{\hbar}f(x', x_n) = f(x', \hbar x_n)$.

As before

$$\|\pi_0^{\partial}(f \oplus K)\| = \sup\{\|(\eta_0(f \oplus K))g\|_{\infty, 0} \mid \|g\|_{\infty} \leq 1\}.$$

Plugging in the definitions of \tilde{f} and \tilde{K} (omitting the cut-off functions) we get

$$\eta_{\hbar}(\tilde{f})g(m_1, m_2, \hbar, b) = \frac{1}{\hbar^{n-1}} \int_{[0, a_{\hbar}/\hbar]} f\left(m_1, \hbar b, -\frac{\exp^{-1}(m_1, m)}{\hbar}, b - a\right)g(m, m_2, \hbar, a) dm da,$$

$$\eta_{\hbar}(\tilde{K})g(m_1, m_2, \hbar, b) = \frac{1}{\hbar^{n-1}} \int_{[0, a_{\hbar}/\hbar]} K\left(m_1, b, -\frac{\exp^{-1}(m_1, m)}{\hbar}, a\right) g(m, m_2, \hbar, a) dm da.$$

Using dominated convergence and the fact that g for small \hbar looks like

$$g_0\left(m, -\frac{\exp^{-1}(m, m_2)}{\hbar}, \hbar, a\right), \quad g_0 \in C_c^\infty(T\partial X \times [0, 1] \times [0, \infty[),$$

we get

$$\lim_{\hbar \rightarrow 0} \|\eta_{\hbar}(\tilde{f} + \tilde{K})g\|_{\infty, \hbar} = \|\eta_0(f \oplus K)g\|_{\infty, 0},$$

and (2.22) follows. \square

Lemmas 2.25 and 2.26 imply that $\liminf_{\hbar \rightarrow 0} \|\varphi_{\hbar}(a)\| \geq \|\varphi_0(a)\|$, i.e., we have

2.27. Theorem. $(C_r^*(T^-X), \{C_r^*(T^-X)(\hbar), \varphi_{\hbar}\}_{\hbar \in [0, 1]})$ is lower semi-continuous in 0.

This finishes the proof of Theorem 2.17.

3. K -theory of the symbol algebra $C_r^*(T^-X)$

$C_c^\infty(TX^\circ)$ with the fiberwise convolution product is a $*$ -ideal of $C_{lc}^\infty(T^-X)$. After completion, $C_r^*(TX^\circ)$ becomes a C^* -ideal of $C_r^*(T^-X)$, and we have a short exact sequence

$$0 \rightarrow C_r^*(TX^\circ) \rightarrow C_r^*(T^-X) \rightarrow C_r^*(T^-X)/C_r^*(TX^\circ) \rightarrow 0. \quad (3.1)$$

3.1. Proposition. The quotient $Q = C_r^*(T^-X)/C_r^*(TX^\circ)$ is naturally isomorphic to $C_0(T^*\partial X) \otimes \mathfrak{K}_0$ for the ideal \mathfrak{K}_0 of the Toeplitz algebra introduced before Lemma 2.14.

Proof. Define

$$\Psi : C_{lc}^\infty(T^-X) \rightarrow \mathcal{L}(L^2(T^+X|_{\partial X})) \quad \text{by } \Psi(f \oplus K) = \pi_0^\partial(f) + \pi_0^\partial(K)$$

with the maps in (2.10) and (2.11). This is a $*$ -homomorphism with respect to $*$ ', and $C_c^\infty(TX^\circ)$ is in its kernel. We first show that $\ker \Psi = C_r^*(TX^\circ)$. Since $C_r^*(T^-X)$ is the closure of $C_{lc}^\infty(T^-X)$ with respect to the norm

$$\|f \oplus K\| = \max\{\|\pi_0(f)\|, \|\pi_0^\partial(f) + \pi_0^\partial(K)\|\},$$

and $C_r^*(TX^\circ)$ is the closure of $C_c^\infty(TX^\circ)$ with respect to $\|\pi_0(f)\|$, we have $C_r^*(TX^\circ) \subseteq \ker \Psi$.

On the other hand, suppose that $a \in \ker \Psi$; i.e., a is the equivalence class of a Cauchy sequence $(f_k \oplus K_k) \in C_{lc}^\infty(T^-X)$ with $\pi_0^\partial(f_k) + \pi_0^\partial(K_k) \rightarrow 0$. We next note that

$$\begin{aligned} \|\pi_0(f_k)\| &= \sup\{|\hat{f}_k(m, \sigma)| \mid (m, \sigma) \in T^*X\} \quad \text{and} \\ \|\pi_0^\partial(f_k) + \pi_0^\partial(K_k)\| &\geq \sup\{|\hat{f}_k(m, \sigma)| \mid (m, \sigma) \in T^*X|_{\partial X}\}. \end{aligned}$$

Indeed the first inequality follows from the fact that, via fiberwise Fourier transform, $\pi_0(f_k)$ is equivalent to multiplication by $\hat{f}_k(m, \sigma)$. For the second, we observe first that $\|\pi_0^\partial(f_k)\| = \sup\{|\hat{f}_k|\}$ as a consequence of the fact that translation of $\xi = \xi(m, w)$ in the direction of w_n preserves $\|\pi_0^\partial(f_k)\xi\|$ in $L^2(T^+X|_{\partial X})$. On the other hand, $\pi_0^\partial(K_k)\xi = 0$ provided we translate sufficiently far. Hence $\|\pi_0^\partial(f_k) + \pi_0^\partial(K_k)\| \geq \|\pi_0^\partial(f_k)\|$.

We conclude that the fiberwise Fourier transforms \hat{f}_k tend to zero uniformly on $T^*X|_{\partial X}$. Hence the Cauchy sequence (f_k) may be replaced by an equivalent Cauchy sequence (g_k) with $g_k \in C_c^\infty(TX^\circ)$. We conclude that $\pi_0^\partial(K_k) \rightarrow 0$ so that $(K_k) \sim 0$, and therefore $\ker \Psi \subseteq C_r^*(TX^\circ)$.

Hence Ψ descends to an injective C^* -morphism on Q ; in particular, it has closed range.

Now we observe that we have a natural identification of $TX|_{\partial X}$ with $T\partial X \times \mathbb{R}$ and consequently of $T^+X|_{\partial X}$ with $T\partial X \times \mathbb{R}_+$. Hence $\mathcal{L}(L^2(T^+X|_{\partial X})) \cong \mathcal{L}(L^2(T\partial X) \otimes L^2(\mathbb{R}_+))$.

Suppose that, at the boundary, $f \in C_c^\infty(TX)$ is of the form $f(x', 0, v', v_n) = g(x', v')h(v_n)$ with $g \in C_c^\infty(T\partial X)$ and $h \in C_c^\infty(\mathbb{R})$. Then $\pi_0^\partial(f) = \pi_0^{\partial,0}(g) \otimes \pi_0^{\partial,n}(h)$, where $\pi_0^{\partial,0}$ is the convolution operator by g , acting on $L^2(T\partial X)$, while $\pi_0^{\partial,n}(h)$ is the operator of half-convolution acting on $L^2(\mathbb{R}_+)$ (note that $\mathbb{R}_+ \cong T^+\mathbb{R}_+|_{\{0\}}$). Via Fourier transform, the operator $\pi_0^{\partial,0}(g)$ is unitarily equivalent to multiplication by $\hat{g} \in C_0(T^*X)$, while, according to Lemma 2.14, $\pi_0^{\partial,n}(h)$ is unitarily equivalent to a Toeplitz operator in \mathfrak{T}_0 . The closure of the image of the span of the pure tensors thus gives us $C_0(T^*\partial X) \otimes \mathfrak{T}_0$.

We know already from Lemma 2.15 that—via the Fourier transform—the image of $C_c^\infty(T\partial X \times \mathbb{R}_+ \times \mathbb{R}_+)$ can also be identified with a subset of $C_0(T^*\partial X) \otimes \mathcal{K} \subseteq C_0(T^*\partial X) \otimes \mathfrak{T}_0$. This completes the argument. \square

3.2. Theorem. *Via fiberwise Fourier transform $C_r^*(TX^\circ)$ can be identified with $C_0(T^*X^\circ)$ and the inclusion $C_0(T^*X^\circ) \cong C_r^*(TX^\circ) \hookrightarrow C_r^*(T^-X)$ induces an isomorphism of K -groups*

$$K_i(C_r^*(T^-X)) \cong K_i(C_0(T^*X^\circ)), \quad i = 0, 1.$$

Proof. It is well known (or easily checked) that $K_i(\mathfrak{T}_0) = 0$, $i = 0, 1$. Thus it follows from the Künneth formula that $K_i(C_0(T^*\partial X) \otimes \mathfrak{T}_0) = 0$, $i = 0, 1$. The result now is a consequence of (3.1) and the associated six-term exact sequence. \square

Acknowledgment

We thank the referee for his careful reading of the article and for many useful remarks.

References

- [1] B. Ammann, R. Lauter, V. Nistor, Algebras of pseudodifferential operators on complete manifolds, *Electron. Res. Announc. Amer. Math. Soc.* 9 (2003) 80–87.
- [2] C. Anantharaman-Delaroche, J. Renault, *Amenable Groupoids*, Monogr. Enseign. Math., vol. 36, Enseignement Math., Geneva, 2000.
- [3] L. Boutet de Monvel, Boundary problems for pseudo-differential operators, *Acta Math.* 126 (1–2) (1971) 11–51.
- [4] L. Coburn, The C^* -algebra generated by an isometry, *Bull. Amer. Math. Soc.* 73 (1967) 722–726.
- [5] L.A. Coburn, The C^* -algebra generated by an isometry. II, *Trans. Amer. Math. Soc.* 137 (1969) 211–217.
- [6] A. Connes, *Noncommutative Geometry*, Academic Press, San Diego, CA, 1994.
- [7] J. Dixmier, *Les C^* -Algèbres et Leurs Représentations*, Cah. Sci., vol. 29, Gauthier–Villars, Paris, 1969.

- [8] R. Douglas, Banach Algebra Techniques in Operator Theory, second ed., Grad. Texts in Math., vol. 179, Springer, New York, 1998.
- [9] B.V. Fedosov, Index theorems, in: Partial Differential Equations, vol. VIII, in: Encyclopaedia Math. Sci., vol. 65, Springer, Berlin, 1996, pp. 155–251.
- [10] N.P. Landsman, B. Ramazan, Quantization of Poisson algebras associated to Lie algebroids, in: Groupoids in Analysis, Geometry, and Physics, Boulder, CO, 1999, in: Contemp. Math., vol. 282, Amer. Math. Soc., Providence, RI, 2001, pp. 159–192.
- [11] N.P. Landsman, Quantization and the tangent groupoid, in: Operator Algebras and Mathematical Physics, Constanța, 2001, Theta, Bucharest, 2003, pp. 251–265.
- [12] S. Melo, R. Nest, E. Schrohe, C^* -structure and K -theory of Boutet de Monvel's algebra, J. Reine Angew. Math. 561 (2003) 145–175.
- [13] R.B. Melrose, Pseudodifferential operators, corners and singular limits, in: Proceedings of the International Congress of Mathematicians, vols. I, II, Kyoto, 1990, Math. Soc. Japan, Tokyo, 1991, pp. 217–234.
- [14] B. Monthubert, F. Pierrot, Indice analytique et groupoides de Lie, C. R. Math. Acad. Sci. Paris 325 (1997) 193–198.
- [15] R. Nest, B. Tsygan, Algebraic index theorem, Comm. Math. Phys. 172 (2) (1995) 223–262.
- [16] R. Nest, B. Tsygan, Algebraic index theorem for families, Adv. Math. 113 (2) (1995) 151–205.
- [17] V. Nistor, A. Weinstein, P. Xu, Pseudodifferential operators on differential groupoids, Pacific J. Math. 189 (1) (1999) 117–152.
- [18] S. Rempel, B.-W. Schulze, Index Theory of Elliptic Boundary Problems, North Oxford Academic, London, 1985.
- [19] J. Renault, A Groupoid Approach to C^* -Algebras, Lectures Notes in Math., vol. 793, Springer, Berlin, 1980.
- [20] M.A. Rieffel, Deformation quantization for actions of \mathbb{R}^d , Mem. Amer. Math. Soc. 106 (1993).